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ORIGINAL PAPER

The dielectric response with respect to the weight distribution of relaxation times

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Abstract The subjects of this paper are the analytical and partly numerical calculations concerning the problem how the dielectric response in complex solid dielectric materials depends on a statistical distribution of relaxation times.

Keywords Electric permittivity · Relaxation time · Fubini Theorem · Lebesgue measure

1 Introduction

Apart from the simplest case of Debye relaxation, τ cannot be interpreted as $\frac{1}{\omega_{\max}}$, where $\omega_{\max}$ is the measurable loss-peak frequency [1–4]. In complex, solid materials there exists a continuous or quasi-continuous statistical distribution of relaxation times across different atoms, clusters, or degrees of freedom [1]. Hence the measurable time parameter should be regarded as the mean, effective value of τ according to its statistical weight distribution [1,2]. Then with the assumption of additive contributions we can write the following expression for the time domain dielectric response function [1]:

$$f(t) = -\frac{d\Phi}{dt}, \tag{1}$$

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where

$$\Phi(t) = \int_0^\infty h(\tau) \exp\left(-\frac{t}{\tau}\right) d\tau. \tag{2}$$

To obtain the normalised, complex electric permittivity, which is one of the most important quantity characterizing dielectric materials, we should calculate the Fourier Transform [3,4]:

$$\frac{\epsilon^* - \epsilon_\infty}{\epsilon_0 - \epsilon_\infty} = -\int_0^\infty e^{-i\omega t} \frac{d\Phi}{dt} dt, \tag{3}$$

where $\epsilon^*(\omega) = \epsilon'(\omega) - i\epsilon''(\omega)$ is the complex electric permittivity (ϵ'—electric permittivity, ϵ''—dielectric loss), ϵ_0, ϵ_∞ are electric permittivities for $\omega \to 0$ and $\omega \to \infty$.

2 Theoretical results

Let $h : [0, \infty) \to [0, \infty)$ be the Lebesgue-measurable function such as:

1. $\int_0^\infty h(\tau)\, d\tau = 1$ (the normalizing condition),
2. $\int_0^\infty \tau h(\tau)\, d\tau = \frac{1}{\omega_{\max}}$—mean value of τ according to its statistical weight distribution h, see [1],
3. there exists the function $\Phi(t) = \int_0^\infty h(\tau) e^{-\frac{t}{\tau}}\, d\tau$ for any $t \in [0.\infty)$ see [1].

Then we have:

$$\Phi'(t) = \frac{d}{dt}\int_0^\infty h(\tau)e^{-\frac{t}{\tau}}\, d\tau = \int_0^\infty h(\tau)\frac{d}{dt}(e^{-\frac{t}{\tau}})\, d\tau = -\int_0^\infty \frac{h(\tau)}{\tau} e^{-\frac{t}{\tau}}\, d\tau. \tag{4}$$

Taking into account Eqs. 3 and 4 we obtain:

$$\frac{\epsilon^*(\omega) - \epsilon_\infty}{\epsilon_0 - \epsilon_\infty} = \int_0^\infty \left(\int_0^\infty \frac{h(\tau)}{\tau} e^{-\frac{t}{\tau}}\, d\tau \right) dt.$$

Let us notice that:

$$\int_0^\infty d\tau \int_0^\infty dt \left| e^{-i\omega t} \frac{h(\tau)}{\tau} e^{-\frac{t}{\tau}} \right| = \int_0^\infty d\tau \int_0^\infty dt \frac{h(\tau)}{\tau} e^{-\frac{t}{\tau}}$$
$$= \int_0^\infty d\tau \frac{h(\tau)}{\tau}(-\tau)\left[e^{-\frac{t}{\tau}}\right]_{t=0}^{t\to\infty} = \int_0^\infty h(\tau)\, d\tau = 1. \tag{5}$$

Hence, if we recall the Fubini theorem [5] we have:

$$\int\limits_{[0,\infty]\times[0,\infty]} e^{-i\omega t}\frac{h(\tau)}{\tau}e^{-\frac{t}{\tau}}d(t\times\tau) = \int\limits_0^\infty e^{-i\omega t}\left[\int\limits_0^\infty \frac{h(\tau)}{\tau}e^{-\frac{t}{\tau}}\,d\tau\right]dt$$
$$= \int\limits_0^\infty d\tau \int\limits_0^\infty e^{-i\omega t}\frac{h(\tau)}{\tau}e^{-\frac{t}{\tau}}\,dt. \tag{6}$$

From Eqs. 5 and 6 we receive:

$$\frac{\epsilon^*(\omega)-\epsilon_\infty}{\epsilon_0-\epsilon_\infty} = \int\limits_0^\infty d\tau \int\limits_0^\infty e^{-i\omega t}\frac{h(\tau)}{\tau}e^{-\frac{t}{\tau}}\,dt = \int\limits_0^\infty d\tau\frac{h(\tau)}{\tau}\int\limits_0^\infty e^{-\left(i\omega+\frac{1}{\tau}\right)t}dt$$
$$= \int\limits_0^\infty d\tau\frac{h(\tau)}{\tau}\left(\frac{-1}{i\omega+\frac{1}{\tau}}\right)\left[e^{-\left(i\omega+\frac{1}{\tau}\right)t}\right]_{t=0}^{t\to\infty} = \int\limits_0^\infty \frac{h(\tau)}{1+i\omega\tau}\,d\tau$$
$$= \int\limits_0^\infty \frac{h(\tau)}{1+\omega^2\tau^2}\,d\tau - i\omega\int\limits_0^\infty \frac{\tau h(\tau)}{1+\omega^2\tau^2}\,d\tau. \tag{7}$$

Hence, keeping in mind that $\epsilon^*(\omega) = \epsilon^{'}(\omega) - i\epsilon^{''}(\omega)$ we obtain the following formulae for the normalized dielectric absorption and dispersion:

$$\frac{\epsilon^{''}(\omega)}{\epsilon_0-\epsilon_\infty} = \omega\int\limits_0^\infty \frac{\tau h(\tau)}{1+\omega^2\tau^2}\,d\tau. \tag{8}$$

$$\frac{\epsilon'-\epsilon_\infty}{\epsilon_0-\epsilon_\infty} = \int\limits_0^\infty \frac{h(\tau)}{1+\omega^2\tau^2}\,d\tau. \tag{9}$$

The Eqs. 8 and 9 give the expressions for dielectric absorption and dispersion from the point of view of the statistical distribution of relaxation times. Such an approach is novel and this problem has never been considered this way.

2.1 Imaginary part of electric permittivity

In this section we will calculate imaginary part of the electric permittivity which has the meaning of the dielectric loss.

We adopt the following assumptions on function $h : \mathbb{R} \to [0,\infty)$:

- there exists meromorphic function $H : \mathbb{C} \to \mathbb{C}$ such that $h = H \restriction_{\mathbb{R}}$ and h is analytic function on $\mathbb{R}$,
- $\forall x \in \mathbb{R}\;\; h(-x) = -h(x)$,

- $\int_0^\infty h(x)\,dx = 1.$

then the function

$$\frac{xh(x)}{1+\omega^2x^2}$$

is even and then

$$\int_0^\infty \frac{xh(x)}{1+\omega^2x^2}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{xh(x)}{1+\omega^2x^2}.$$

2.1.1 Rational simple case

Let us consider the function $h : \mathbb{R} \to [0,\infty)$ given by the following formula

$$h(x) = c_{mn}\frac{x^m}{1+x^n},$$

where $m, n \in \mathbb{N}$ and $m \equiv 1 \mod 2$ and $n \equiv 0 \mod 2$, $0 < m+1 < n$ and $c_{mn} \in (0,\infty)$ is a positive real number such that $\int_0^\infty h(x)\,dx = 1$.

Let $R > 0$ be any positive real number and let us define contour in the complex plane $C_R \subset \mathbb{C}$ as follows:

$$C_R = \{z \in \mathbb{C} : |z| \le R \wedge Imz = 0\} \cup \{Re^{it} \in \mathbb{C} : t \in (0,\pi)\}.$$

Here let $I_R = \{z \in \mathbb{C} : |z| \le R \wedge Imz = 0\}$ $\Gamma_R = \{Re^{it} \in \mathbb{C} : t \in (0,\pi)\}$ and observe that

$$\left|\frac{zh(z)}{1+\omega^2z^2}dz\right| = \left|\int_{\Gamma_R}\frac{z^m}{1+z^n}\cdot\frac{z}{1+\omega^2z^2}dz\right|$$
$$= \left|\int_0^\pi \frac{R^{m+1}e^{it}}{(1+R^ne^{int})(1+\omega^2R^2e^{i2t})}Rie^{it}\,dt\right| \le A\cdot R^{m+2-n-2},$$

where A is some fixed positive real whenever R is sufficiently large. Thus we have immediately

$$0 \le \lim_{R\to\infty}\left|\int_{\Gamma_R}\frac{z^m}{1+z^n}\cdot\frac{z}{1+\omega^2z^2}dz\right| \le \lim_{R\to\infty}\frac{A}{R^{n-m}} = 0.$$

As $n - m > 0$ then

$$\int_{-\infty}^{\infty} \frac{xh(x)dx}{1+\omega^2x^2} = \lim_{R\to\infty} \int_{-R}^{R} \frac{xh(x)dx}{1+\omega^2x^2} = \lim_{R\to\infty} \left(\int_{I_R} \frac{zh(z)}{1+\omega^2z^2}dz + \int_{\Gamma_R} \frac{zh(z)}{1+\omega^2z^2}dz \right)$$

$$= \lim_{R\to\infty} \int_{C_R} \frac{zh(z)}{1+\omega^2z^2}dz = \int_{C} \frac{zh(z)}{1+\omega^2z^2}dz,$$

where C is the contour which arounds the upper complex halfplane.

Now let us consider the set of all roots of the equation $z^n + 1 = 0$ with the positive imaginary part

$$\left\{ z_k \in \mathbb{C} : z_k = e^{\frac{\pi+2k\pi}{n}i} \wedge k \in \left\{0, \ldots, \frac{n}{2} - 1\right\}\right\}.$$

The rest of the roots has a negative imaginary part (the are in the open down complex halfplane) and let $\{z_w, \overline{z_w}\}$ is the set of all roots of the $1+\omega^2z^2 = 0$ where $Imz_w > 0$. Let r be any positive real number such that $r < min\{|z_i - z_j| : i \neq j \wedge i, j \in \{0, \ldots, \frac{n}{2} - 1\}\}$ then let define circles C_k for $k \in \left\{0, \ldots, \frac{n}{2} - 1\right\}$ as follows:

$$\forall k \in \left\{0, \ldots, \frac{n}{2} - 1\right\} \quad C_k = \{z \in \mathbb{C} : \exists t \in [0, 2\pi) \;\; z = z_k + re^{it}\}.$$

Analogously let us define

$$C_w = \{z \in \mathbb{C} : \exists t \in [0, 2\pi) \;\; z = z_w + se^{it}\},$$

where $s = \frac{Imz_w}{2}$.

Then by residuum theorem we have

$$\int_{-\infty}^{\infty} \frac{xh(x)}{1+\omega^2x^2}dx = \int_{C} \frac{zh(z)}{1+\omega^2z^2}dz$$

$$= \sum_{k=0}^{\frac{n}{2}-1} \int_{C_k} \frac{z^{m+1}}{(1+\omega^2z^2)\prod_{i\neq k}(z-z_i)} \frac{dz}{z-z_k}$$

$$+ \int_{C_w} \frac{z^{m+1}}{(z-\overline{z_w})(1+z^n)} \frac{dz}{z-z_w}$$

$$= 2\pi i \sum_{k=0}^{\frac{n}{2}-1} \frac{z_k^{m+1}}{(1+\omega^2z_k^2)\prod_{i\neq k}(z_k-z_i)} + 2\pi i \frac{z_w^{m+1}}{(z_w-\overline{z_w})(1+z_w^n)}.$$

2.1.2 Rational case

Let us consider the function $h : \mathbb{R} \to [0, \infty)$ given by the following formula

$$h(x) = B\frac{x^{2s-1}\prod_{k=1}^{m}(x^2 + a_k^2)}{\prod_{k=1}^{n}(x^2 + b_k^2)},$$

where

1. $m, n, s \in \mathbb{N}$ and $0 < 2m + 2s + 1 < 2n$,
2. $\{a_k \in \mathbb{R} : k \in \{1, \ldots, m\}\}$ is the subset of the real line,
3. $\{b_k \in \mathbb{R} : k \in \{1, \ldots, n\}\}$ is the subset of the real line with pairwise different elements,
4. $B \in (0, \infty)$ is a positive real number such that $\int_0^\infty h(x)\,dx = 1$.

Let $R > 0$ be any positive real number and let us define contour in the complex plane $C_R \subset \mathbb{C}$ as follows:

$$C_R = \{z \in \mathbb{C} : |z| \leq R \wedge Imz = 0\} \cup \{Re^{it} \in \mathbb{C} : t \in (0, \pi)\}.$$

Here let $I_R = \{z \in \mathbb{C} : |z| \leq R \wedge Imz = 0\}$ $\Gamma_R = \{Re^{it} \in \mathbb{C} : t \in (0, \pi)\}$ and observe that

$$\begin{aligned}\left|\frac{zh(z)}{1+\omega^2 z^2}dz\right| &= \left|\int_{\Gamma_R} \frac{z^m}{1+z^n} \cdot \frac{z}{1+\omega^2 z^2}dz\right| \\ &= \left|\int_0^\pi \frac{R^{m+1}e^{it}}{(1+R^n e^{int})(1+\omega^2 R^2 e^{i2t})} Rie^{it}\,dt\right| \leq A \cdot R^{m+2-n-2},\end{aligned}$$

where A is some fixed positive real whenever R is sufficiently large. Thus we have immediately

$$0 \leq \lim_{R\to\infty}\left|\int_{\Gamma_R} \frac{z^m}{1+z^n} \cdot \frac{z}{1+\omega^2 z^2}dz\right| \leq \lim_{R\to\infty}\frac{A}{R^{n-m}} = 0$$

because $n - m > 0$.

Then we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{xh(x)dx}{1+\omega^2 x^2} &= \lim_{R\to\infty}\int_{-R}^{R} \frac{xh(x)dx}{1+\omega^2 x^2} = \lim_{R\to\infty}\left(\int_{I_R} \frac{zh(z)}{1+\omega^2 z^2}dz + \int_{\Gamma_R} \frac{zh(z)}{1+\omega^2 z^2}dz\right) \\ &= \lim_{R\to\infty}\int_{C_R} \frac{zh(z)}{1+\omega^2 z^2}dz = \int_C \frac{zh(z)}{1+\omega^2 z^2}dz,\end{aligned}$$

where C is the contour which arounds the upper complex halfplane.

Now let us consider the set of all roots for equations $z^2 + b_j^2 = 0$ where $j \in \{1, \ldots, n\}$ with the positive imaginary part

$$\{z_j \in \mathbb{C} : z_j = i|b_j| \wedge 1 \in \{1, \ldots, n\}.$$

The rest of the roots has a negative imaginary part of course and let observe that

$$\left\{\frac{i}{\omega}, -\frac{i}{\omega}\right\}$$

are the roots of the $1 + \omega^2 z^2 = 0$ and let $z_w = \frac{i}{\omega}$.

Let r be any positive real number such that $r < min\{|z_i - z_j| : \ i \neq j \wedge i, j \in \{0, \ldots, \frac{n}{2} - 1\}\}$ then let us define circles C_k for $k \in \left\{0, \ldots, \frac{n}{2} - 1\right\}$ as follows:

$$\forall k \in \left\{0, \ldots, \frac{n}{2} - 1\right\} \quad C_k = \{z \in \mathbb{C} : \ \exists t \in [0, 2\pi) \ \ z = z_k + re^{it}\}.$$

Analogously let us define

$$C_w = \{z \in \mathbb{C} : \ \exists t \in [0, 2\pi) \ \ z = z_w + se^{it}\},$$

where $s = \frac{Imz_w}{2}$.

Let $h_1(z) = \frac{1}{B}h(z)$, then by residuum theorem we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{xh_1(x)}{1+\omega^2x^2}dx &= \int_C \frac{zh_1(z)}{1+\omega^2z^2}dz = \int_C \frac{z^{2s}\prod_{k=1}^m(z^2+a_k^2)}{\prod_{k=1}^n(z^2+b_k^2)}\frac{1}{1+\omega^2z^2}dz \\
&= \sum_{j=1}^n \int_{C_j} \frac{z^{2s}\prod_{k=1}^m(z^2+a_k^2)}{\prod_{k=1\wedge j\neq k}^n(z^2+b_k^2)}\frac{1}{1+\omega^2z^2}\frac{1}{z-\overline{z_j}}\frac{dz}{z-z_j} \\
&\quad + \int_{C_w} \frac{zh_1(z)}{z-\overline{z_w}}\frac{dz}{z-z_w} \\
&= 2\pi i \sum_{j=1}^n \frac{z_j^{2s}\prod_{k=1}^m(z_j^2+a_k^2)}{\prod_{k=1\wedge j\neq k}^n(z_j^2+b_k^2)}\frac{1}{1+\omega^2z_j^2}\frac{1}{z_j-\overline{z_j}} + 2\pi i\frac{z_wh_1(z_w)}{z_w-\overline{z_w}}
\end{aligned}$$

And finally we have

$$\int_{-\infty}^{\infty} \frac{xh_1(x)}{1+\omega^2x^2}dx = \pi\sum_{j=1}^{n}(-1)^s \frac{b_j^{2s}\prod_{k=1}^{m}(a_k^2-b_j^2)}{\prod_{k=1\wedge j\neq k}^{n}(b_k^2-b_j^2)}\frac{1}{1-\omega^2b_j^2}\frac{1}{b_j}$$
$$+\pi(-1)^s\frac{1}{\omega^s}\frac{\prod_{k=1}^{m}(a_k^2-\frac{1}{\omega^2})}{\prod_{k=1}^{n}(b_k^2-\frac{1}{\omega^2})}$$
$$= (-1)^s\pi\left(\sum_{j=1}^{n}\frac{\prod_{k=1}^{m}(a_k^2-b_j^2)}{\prod_{k=1\wedge j\neq k}^{n}(b_k^2-b_j^2)}\frac{b_j^{2s-1}}{1-\omega^2b_j^2}+\frac{1}{\omega^s}\frac{\prod_{k=1}^{m}(a_k^2-\frac{1}{\omega^2})}{\prod_{k=1}^{n}(b_k^2-\frac{1}{\omega^2})}\right).$$

If we assume that $m, n, s \in \mathbb{N}$ and $0 < 2m + 2s + 3 < 2n$ then the expectation value $E[X] = \int_0^\infty xh(x)$ exists. Using the above technique we have

$$E[X] = A(-1)^s\sum_{j=1}^{n}b_j^{2s-1}\frac{\prod_{k=1}^{m}(a_k^2-b_j^2)}{\prod_{k=1\wedge k\neq j}^{n}(b_k^2-b_j^2)}.$$

2.1.3 Non-rational case

Let us consider function $h, f : \mathbb{C} \to \mathbb{C}$ given by the following formula

$$f(z) = \frac{h(z)z^\lambda}{\prod_{j=1}^{n}((z+\alpha_j)^2+\beta_j^2)} \quad \text{and} \quad g(z) = A\cdot f(z),$$

where $n \in \mathbb{N}$ and $A \in (0,\infty)$ is a positive real number such that $\int_0^\infty h(x)\,dx = 1$ and let assume that $\lambda + deg(h) + 2 < 2n$.

Let $0 < r < R$ be any positive real number and let us define contour in the complex plane $C_R \subset \mathbb{C}$ as follows:

$$C_R = I_+ \cup \Gamma_R \cup I_- \cup \Gamma_r,$$

where

- $I_+ = \{t \in \mathbb{C} : t \in (r, R)\}$,
- $\Gamma_R = \{Re^{it} \in \mathbb{C} : t \in (0, 2\pi)\}$,
- $I_- = \{R - t \in \mathbb{C} : t \in (0, R - r)\}$,
- $\Gamma_r = \{re^{i(2\pi - t)} \in \mathbb{C} : t \in (0, 2\pi)\}$.

Here let $m = deg(h)$ and $\delta = 2n - 2 - m - \lambda > 0$ then

$$\exists R_0 > 0 \exists M > 0 \quad \forall z \in \mathbb{C} \quad R_0 < |z| \to |f(Z)| \le M\left|\frac{z^{\lambda+m}}{z^{2n}}\right| = \frac{M}{|z|^{2+\delta}}.$$

Hence we are getting

$$\left|\int_{\Gamma_R} zf(z)\,dz\right| = \left|\int_0^{2\pi} Re^{it} f(Re^{it}) iRe^{it}\,dt\right| \le \int_0^{2\pi} |f(Re^{it})| R^2\,dt$$
$$\le \int_0^{2\pi} \frac{M}{R^{2+\delta}} R^2\,dt = \frac{2\pi M}{R^{\delta}}.$$

Here we are getting immediately

$$\lim_{R\to\infty} \int_{\Gamma_R} zf(z)\,dz = 0.$$

Moreover, doing analogously we have

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz = 0 \wedge \lim_{R\to\infty} \int_{\Gamma_R} \frac{zf(z)}{1+\omega^2 z^2}\,dz = 0.$$

Now let $R \in \{R_0, z \cdot R_0, \frac{z}{1+\omega^2 z^2} \cdot R_0\}$ where

$$R_0(z) = \frac{h(z)}{\prod_{j=1}^{n}((z+\alpha_j^2)^2+\beta_j^2)}.$$

Let us observe that we have

$$\lim_{\substack{R\to\infty\\ r\to 0}} \int_{C_R} z^{\lambda} R(z)\,dz = \int_0^{\infty} x^{\lambda} R(x)\,dx + \lim_{R\to\infty} \int_{\Gamma_R} z^{\lambda} R(z)\,dz$$
$$+(-1)\int_0^{\infty} t^{\lambda} e^{2\pi i\lambda} R(t)\,dt + \lim_{r\to 0}(-1)\int_{\Gamma_r} z^{\lambda} R(z)\,dz$$
$$= \int_0^{\infty} x^{\lambda} R(x)\,dx - e^{2\pi i\lambda}\int_0^{\infty} x^{\lambda} R(x)\,dx$$
$$= (1-e^{2\pi i\lambda})\int_0^{\infty} x^{\lambda} R(x)\,dx.$$

Let us observe that

$$(1-e^{2\pi i\lambda})^{-1} = \frac{1}{2} + \frac{i}{2}\frac{1+\cos 2\pi\lambda}{\sin 2\pi\lambda}$$

then we obtain

$$\int_0^\infty x^\lambda R(x)\,dx = \left(\frac{1}{2} + \frac{i}{2}\frac{1+\cos 2\pi\lambda}{\sin 2\pi\lambda}\right)\int_C z^\lambda R(z)\,dz,$$

where

$$\int_C z^\lambda R(z)\,dz = \lim_{\substack{R\to\infty \\ r\to 0}} \int_{C_R} z^\lambda R(z)\,dz.$$

Let $Z = \{z \in \mathbb{C} :\ z$ is zero of $\prod_{j=1}^n((z+\alpha_j)^2+\beta_j^2)\}$ $Z_h = \{z \in \mathbb{C} :\ \ z\cdot h(z) = 0\}$ and let $Z_\omega = \{z \in \mathbb{C} :\ \ 1+\omega^2 z^2 = 0\}$ then by our assumption these sets are pairwise disjoint i.e. $Z \cap Z_h = \emptyset$ $Z \cap Z_\omega = \emptyset$ and $Z_h \cap Z_\omega = \emptyset$. Let $Z_+ = \{z \in Z :\ \ Imz > 0\}$ and $Z_- = \{z \in \mathbb{C} :\ \ Imz < 0\}$ then $Z = Z_+ \cup Z_-$ of course and let us enumerate of them as follows

$$Z_+ = \{z_k^+ = -\alpha_k + \beta_k i \in \mathbb{C} :\ \ k \in \{1, \dots, n\}\},$$

$$Z_- = \{z_k^- = -\alpha_k - \beta_k i \in \mathbb{C} :\ \ k \in \{1, \dots, n\}\}$$

and $Z_\omega = \{z_\omega^+, z_\omega^-\} = \{\frac{i}{\omega}, -\frac{i}{\omega}\}$ of course and let us observe that there exists the set of the positive real number $r > 0$ (radius of the circle) such that the family

$$\{C_k^+, C_k^- :\ \ k \in \{1, \dots, n\}\} \cup \{C_\omega^+, C_\omega^-\}$$

is pairwise disjoint where for any $k \in \{1, \dots, n\}$

$$C_k^+ = \{z_k^+ + re^{it} \in \mathbb{C} :\ t \in [0, 2\pi)\}, \quad C_k^- = \{z_k^- + re^{it} \in \mathbb{C} :\ t \in [0, 2\pi)\},$$

and $C_\omega^+ = \{z_\omega^+ + re^{it} \in \mathbb{C} :\ \ t \in [0, 2\pi)\}$, $C_\omega^- = \{z_\omega^- + re^{it} \in \mathbb{C} :\ \ t \in [0, 2\pi)\}$.

case *1)* $R(z) = R_0(z)$

By residuum Theorem we have

$$\int_C z^\lambda R(z)\,dz$$

$$= \sum_{k=1}^n \left(\int_{C_k^+} \frac{z^\lambda h(z)}{\prod_{\substack{j=1 \\ j\neq k}}^n((z+\alpha_j)^2+\beta_j^2)} \cdot \frac{1}{z-(-\alpha_k-\beta_k i)} \cdot \frac{dz}{z-(-\alpha_k+\beta_k i)} \right.$$

$$
\left. + \int_{C_k^-} \frac{z^\lambda h(z)}{\prod_{\substack{j=1\\ j\neq k}}^{n} ((z+\alpha_j)^2 + \beta_j^2)} \cdot \frac{1}{z - (-\alpha_k + \beta_k i)} \cdot \frac{dz}{z - (-\alpha_k - \beta_k i)} \right)
$$

$$
= 2\pi i \sum_{k=1}^{n} \frac{(-\alpha_k + \beta_k i)^\lambda h(-\alpha_k + \beta_k i)}{\prod_{\substack{j=1\\ j\neq k}}} (\beta_j^2 - \beta_k^2) \frac{1}{2\beta_k i}
$$

$$
+ 2\pi i \sum_{k=1}^{n} \frac{(-\alpha_k - \beta_k i)^\lambda h(-\alpha_k - \beta_k i)}{\prod_{\substack{j=1\\ j\neq k}}^{n} (\beta_j^2 - \beta_k^2)} \frac{1}{-2\beta_k i}
$$

$$
= \pi \sum_{k=1}^{n} \frac{(-\alpha_k + \beta_k i)^\lambda h(-\alpha_k + \beta_k i) - (-\alpha_k - \beta_k i)^\lambda h(-\alpha_k - \beta_k i)}{\beta_k \prod_{\substack{j=1\\ j\neq k}}^{n} (\beta_j^2 - \beta_k^2)}.
$$

case *2)* $R(z) = zR_0(z)$

We can calculate as in the previous case, then we have:

$$
\int_C z^\lambda R(z)\, dz
$$

$$
= \pi \sum_{k=1}^{n} \frac{(-\alpha_k + \beta_k i)^{\lambda+1} h(-\alpha_k + \beta_k i) - (-\alpha_k - \beta_k i)^{\lambda+1} h(-\alpha_k - \beta_k i)}{\beta_k \prod_{\substack{j=1\\ j\neq k}}^{n} (\beta_j^2 - \beta_k^2)}
$$

Case *3)* $R(z) = \frac{zR_0(z)}{1+\omega^2 z^2}$

Similarly we have

$$
\int_C z^\lambda R(z)\, dz = \pi \sum_{k=1}^{n} \frac{1}{\beta_k \prod_{\substack{j=1\\ j\neq k}}^{n} (\beta_j^2 - \beta_k^2)}
$$

$$
\times \left(\frac{(-\alpha_k + \beta_k i)^{\lambda+1} h(-\alpha_k + \beta_k i)}{(1 + \omega^2(-\alpha_k + \beta_k i))^2} \right.
$$

$$
\left. - \frac{(-\alpha_k - \beta_k i)^{\lambda+1} h(-\alpha_k - \beta_k i)}{(1 + \omega^2(-\alpha_k - \beta_k i))^2} \right)
$$

$$
+ \frac{\pi}{\omega} \frac{1}{\prod_{j=1}^{n} \beta_j^2 - \frac{1}{\omega^2}} \left(\left(\frac{i}{\omega} \right)^\lambda h\left(\frac{i}{\omega} \right) - \left(\frac{-i}{\omega} \right)^\lambda h\left(\frac{-i}{\omega} \right) \right).
$$

Probability density As above the positive $g(x) = A \cdot f(x) = A \frac{x^\lambda h(x)}{\prod_{j=1}^n ((x+\alpha_j^2)+\beta_j^2)}$ should be probabilistic distribution then $\int\limits_0^\infty g(x)\,dx = 1$ and then

$$\begin{aligned} 1 &= \int\limits_0^\infty g(x)\,dx = A \int\limits_0^\infty \frac{x^\lambda h(x)}{\prod_{j=1}^n ((x+\alpha_j^2)+\beta_j^2)}\,dx \\ &= \frac{A}{2}\left(1 + i\frac{1+\cos 2\pi\lambda}{\sin 2\pi\lambda}\right)\int\limits_C z^\lambda R_0(z)\,dz \\ &\overset{(\text{case } 1)}{=} \frac{A}{2}\left(1 + i\frac{1+\cos 2\pi\lambda}{\sin 2\pi\lambda}\right) \\ &\quad \times \pi \sum_{k=1}^n \frac{1}{\beta_k \prod_{\substack{j=1\\ j\neq k}}^n (\beta_j^2 - \beta_k^2)} \big((-\alpha_k + i\beta_k)^\lambda h(-\alpha_k + i\beta_k) \\ &\quad - (-\alpha_k - i\beta_k)^\lambda h(-\alpha_k - i\beta_k)\big), \end{aligned}$$

which allows us to compute the constant A.

Expectation value Now we are ready to find expectation value for our non-rational distribution function $g(x)$

$$\begin{aligned} E[X] &= \int\limits_0^\infty x \cdot g(x)\,dx = A \int\limits_0^\infty x \frac{x^\lambda h(x)}{\prod_{j=1}^n ((x+\alpha_j)^2+\beta_j^2)}\,dx \\ &= \frac{A}{2}\left(1 + i\frac{1+\cos 2\pi\lambda}{\sin 2\pi\lambda}\right)\int\limits_C \frac{z^{\lambda+1} h(z)}{\prod_{j=1}^n ((z+\lambda_j)^2+\beta_j^2)} \end{aligned}$$

$$\begin{aligned} &\overset{(\text{case } 2)}{=} \left(1 + i\frac{1+\cos 2\pi\lambda}{\sin 2\pi\lambda}\right)\pi \sum_{k=1}^n \frac{1}{\beta_k \prod_{\substack{j=1\\ j\neq k}}^n (\beta_j^2 - \beta_k^2)} \\ &\quad \times \left((-\alpha_k + i\beta_k)^{\lambda+1} h(-\alpha_k + i\beta_k) - (-\alpha_k - i\beta_k)^{\lambda+1} h(-\alpha_k - i\beta_k)\right). \end{aligned}$$

Dielectric response Here we can write down the formula for dielectric response using our non-rational distribution function. Finally we have

$$\begin{aligned} & Im \frac{\epsilon^* - \epsilon_\infty}{\epsilon_0 - \epsilon_\infty} \\ &\quad = -\omega \int\limits_0^\infty \frac{g(x)\cdot x}{1+\omega^2 x^2}\,dx = -A\omega \int\limits_0^\infty \frac{x^{\lambda+1}}{1+\omega^2 x^2} \cdot \frac{h(x)}{\prod_{j=1}^n ((x+\alpha_j)^2+\beta_j^2)}\,dx \end{aligned}$$

$$= -\frac{A\omega}{2}\left(1 + i\frac{1+\cos 2\pi\lambda}{\sin 2\pi\lambda}\right)\int_C \frac{z^{\lambda+1}h(z)}{(1+\omega^2 z^2)\prod_{j=1}^{n}((z+\lambda_j)^2+\beta_j^2)}$$

$$\stackrel{\text{(case 3)}}{=} -\frac{\pi\omega A}{2}\left(1 + i\frac{1+\cos 2\pi\lambda}{\sin 2\pi\lambda}\right)\left\{\left(\sum_{k=1}^{n}\frac{1}{\prod_{\substack{j=1\\ j\neq k}}^{n}}(\beta_j^2-\beta_k^2)\right.\right.$$

$$\left.\times\left[\frac{(-\alpha_k+i\beta_k)^{\lambda+1}h(-\alpha_k+i\beta_k)}{1+\omega^2(-\alpha_k+i\beta_k)^2} - \frac{(-\alpha_k-i\beta_k)^{\lambda+1}h(-\alpha_k-i\beta_k)}{1+\omega^2(\alpha_k+i\beta_k)^2}\right]\right)$$

$$\left.+\frac{\pi}{\omega}\left(\frac{i}{\omega}\right)^{\lambda+1}\frac{1}{\prod_{j=1}^{n}(\beta_j^2-\frac{1}{\omega^2})}\left[h\left(\frac{i}{\omega}-(-1)^{\lambda+1}h\left(-\frac{i}{\omega}\right)\right)\right]\right\},$$

where $(-1)^{\lambda+1} = \cos(\lambda+1)\pi + i\sin(\lambda+1)\pi$ of course.

3 Numerical results

The numerical calculations were performed with Mathematica package. Here we present some results of the calculations of the normalized dielectric absorption (see Figs. 1 and 2) for two exemplifying functions $h(\tau)$ (see Figs. 3 and 4):

$$h(\tau) = \frac{A\tau}{1+(\alpha\tau)^4} \tag{10}$$

and

$$h(\tau) = Ae^{-\alpha\tau}. \tag{11}$$

These functions were chosen a priori to illustrate our model although we think that they may describe a real distribution of relaxation times in certain materials. We have come across neither theoretical nor experimental data concerning the continuous weight distribution of relaxation times in real materials. The constants A and α appearing in Eqs. 10 and 11 can be easily obtained from conditions (1) and (2).

The frequency dependence of normalized dielectric absorption we show for better clarity in log–log scale. The curve in Fig. 1 reveals the symmetry in relation to the absorption peak, while the second one in Fig. 2 seems to be asymmetrical. If we analyze this dependence in respect of the power-laws [1,4,6–8] we are easily able to make out the linearity for frequencies sufficiently distant from the loss-peak frequency $\omega_{\max}$ in both considered cases (Figs. 1 and 2), although we have the broader range of non-linearity as far as the second absorption curve is concerned.

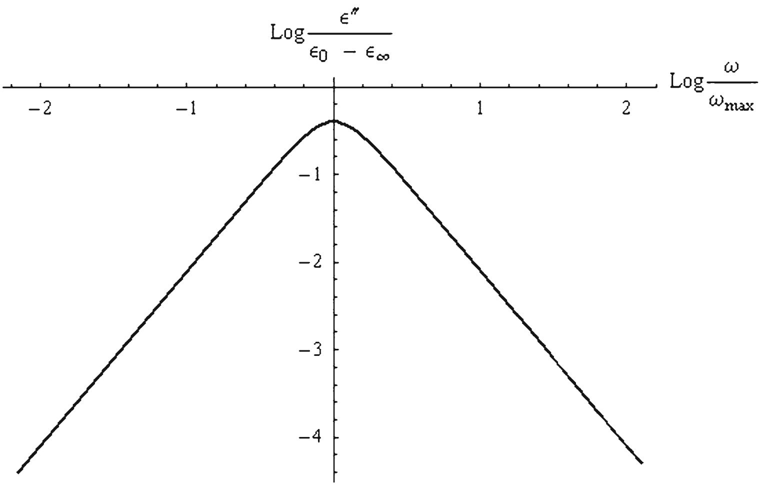

Fig. 1 Results of numerical calculations of normalized dielectric absorption for (10)

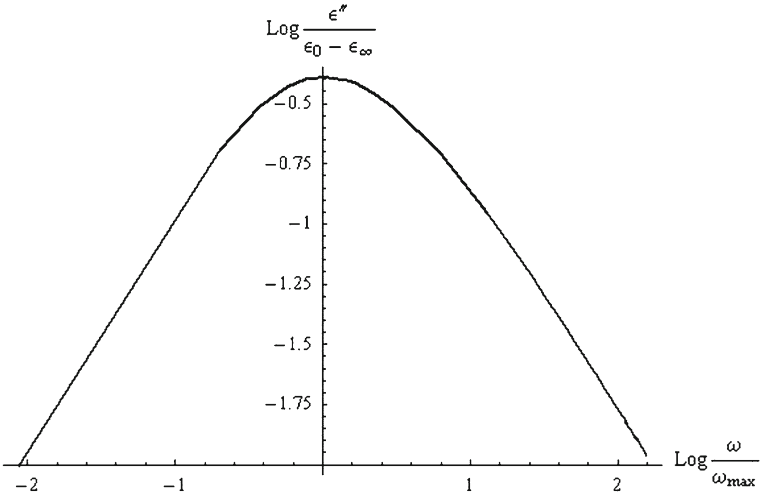

Fig. 2 Results of numerical calculations of normalized dielectric absorption for (11)

4 Conclusions

- We managed to obtain the formulae for the normalized dielectric absorption and dispersion, which are dependent on the distribution of relaxation times.
- We think that a specific type of dielectric response results from a specific class of $h(\tau)$ functions. Such a correspondence is rather difficult to find from the mathematical point of view. It will be the subject of further research.

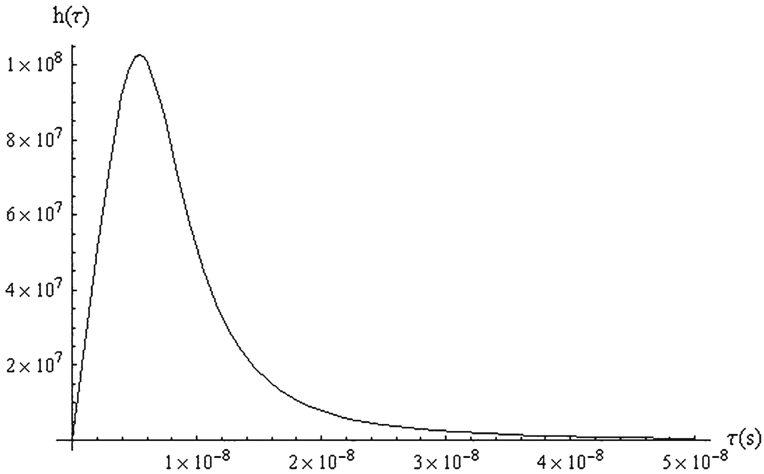

Fig. 3 The exemplifying function of relaxation times distribution (10) for $\omega_{\max} = 10^8\ \mathrm{s}^{-1}$ vs. $\tau(s)$

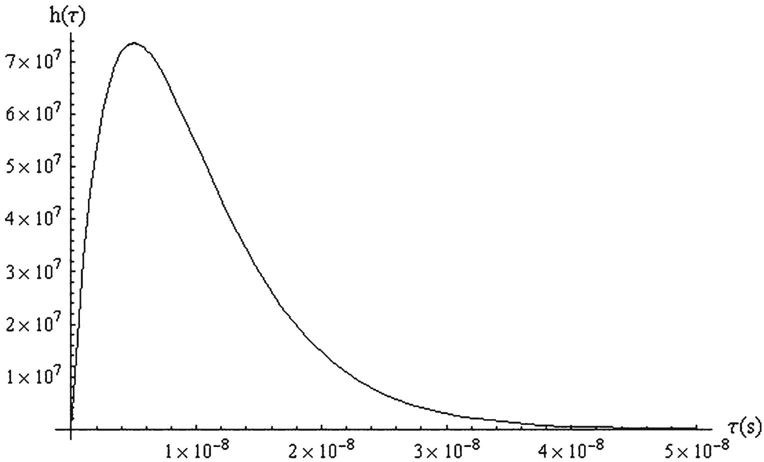

Fig. 4 The exemplifying function of relaxation times distribution (11) for $\omega_{\max} = 10^8\ \mathrm{s}^{-1}$ vs. $\tau(s)$

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